View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 3, 344—350 (1961)

A Third Order Autonomous Differential Equation with Almost Periodic Solutions

R. N. D'HEEDENE

Harvard University, Cambridge, Massachusetts

Submitted by N. Levinson

I. Introduction

The Poincaré-Bendixson Theorem states that solutions of second order autonomous differential equations (and, more generally, systems of two first order equations) which are bounded in phase space and tend to no critical point must tend to a closed cycle as time increases. We approach here the question of whether this result holds for third order equations, and show by an example that the answer is negative. (The question is trivially answered in the negative for systems of three first order equations and for equations of higher order.)

The example consists of constructing a single third order equation

$$\dddot{x} = F(x, \dot{x}, \ddot{x}) \tag{1}$$

such that, aside from three exceptional orbits, each orbit in the phase space approaches an entire surface S, toroidal in shape, as time tends to infinity. For each point p lying on S, S contains the entire orbit γ_p passing through p; γ_p itself is dense in S and almost periodic but not periodic. Note that for a surface to contain such orbits it must, like a torus, have genus higher than unity; for otherwise the ideas of the Poincaré-Bendixson Theorem (i.e. the Jordan Curve Theorem) apply in the surface.

For a proof of the Poincaré-Bendixson Theorem, and other background material, see Coddington and Levinson [1]. Three papers that discuss the existence of a closed cycle for a third order autonomous equation are Friedrichs [2], Levinson [3], and Rauch [4]. Levinson [5] is recommended as a fascinating paper, although not directly related to the present work.

II. Constructions

In constructing our example, it will be convenient to start from the system of three first order equations

$$\dot{x} = y, \qquad \dot{y} = z - x, \qquad \dot{z} = Z(x, y, z). \tag{2}$$

Each solution of (2) is indeed the solution of a single third order equation, namely

$$\dddot{x} = Z(x, \dot{x}, x + \ddot{x}) - \dot{x}. \tag{3}$$

It will also be convenient to introduce coordinates more natural to a torus than x, y, z. Let θ denote the negative of the usual angle of plane polar coordinates so that

$$\cos\theta = x/r, \qquad \sin\theta = -\,y/r, \tag{4}$$

where $r = \sqrt{x^2 + y^2}$. Let R be a positive constant, and define the distance ρ and angle ϕ by

$$\rho = \sqrt{(r - R)^2 + z^2}, \tag{5}$$

$$\cos\phi = (r - R)/\rho, \qquad \sin\phi = z/\rho. \tag{6}$$

Thus the equation $\rho = a < R$ defines a torus whose center line is the circle $r = R$, $z = 0$ and whose cross section is a circle of radius a; the angle ϕ is measured around this circle of radius a. Note that the coordinates θ, ϕ, ρ become ambiguous on the z-axis; we shall never use them there.

It is easy to compute that in terms of θ, ϕ, and ρ, (2) becomes

$$\dot{\theta} = 1 - \frac{\rho\cos\theta\sin\phi}{R + \rho\cos\phi}. \tag{7a}$$

$$\dot{\phi} = \frac{Z}{\rho}\cos\phi + \sin\theta\sin^2\phi, \tag{7b}$$

$$\dot{\rho} = (Z - \rho\sin\theta\cos\phi)\sin\phi. \tag{7c}$$

Let ε_0 be a positive constant small compared to R, and let δ be a real number with $|\delta|$ small compared to ε_0. Let $g(\varepsilon)$ be a smooth function defined for $0 \leqslant \varepsilon \leqslant R$ such that $g(\varepsilon_0) = \delta$ and $g(\varepsilon) = 0$ for ε outside the interval $(\varepsilon_0 - |\delta|, \varepsilon_0 + |\delta|)$. Let $S(\varepsilon)$ denote the toroidal surface

$$S(\varepsilon): \qquad \rho = \varepsilon + g(\varepsilon)\sin^2\phi\sin\theta. \tag{8}$$

We shall use the particular surface

$$S = S(\varepsilon_0): \qquad \rho = \varepsilon_0 + \delta \sin^2 \phi \sin \theta \tag{9}$$

as the surface toward which (almost) all the orbits tend as time increases. The other surfaces $S(\varepsilon)$ will be useful in proving that orbits indeed approach S, for we shall construct Z so that all orbits pierce $S(\varepsilon)$ in the appropriate sense (inward for $\varepsilon > \varepsilon_0$ and outward for $\varepsilon < \varepsilon_0$). The number δ has been introduced to perturb S slightly from a circular cross section, necessary to prevent the periodicity of orbits in S.

Before defining Z, we must introduce two more functions. Let $F(x, y, z)$ be a smooth function which is zero on $S = S(\varepsilon_0)$, negative inside S, positive outside S, and in particular $F = 1$ for $r \leqslant R/2$. Let $G(x, y, z)$ equal $g(\varepsilon)$ if (x, y, z) lies on $S(\varepsilon)$ and let $G = 0$ otherwise. Note that G is non-zero only near (within distance δ) $S = S(\varepsilon_0)$, and is constant on each torus $S(\varepsilon)$.

Define $Z_\varepsilon(x, y, z)$ by

$$Z_\varepsilon = \frac{(\rho + 2G \sin \theta \sin^2 \phi) \sin \theta \cos \phi + G\dot{\theta} \sin \phi \cos \theta}{1 - (2G/\rho) \cos^2 \phi \sin \theta}, \tag{10}$$

where $\dot{\theta}$ is given by (7a). When $G = 0$, this reduces to

$$Z_\varepsilon = \left(\frac{R}{r} - 1\right) y. \tag{11}$$

Let, to finally define the differential equation (2),

$$Z(x, y, z) = \begin{cases} Z_\varepsilon - zF & \text{when} \quad r \geqslant R/2 \\ y - z & \text{when} \quad r \leqslant R/2. \end{cases} \tag{12}$$

It is easy to see that this differential equation satisfies a Lipschitz condition. We now must prove our various claims.

III. Orbits off S

We start by showing that orbits in the phase space tend to S as time increases. To do this, we fill the space with nested surfaces as follows: For $r \geqslant R/2$, use tori of the form (8), where now $0 < \varepsilon < \infty$. For $\varepsilon > R/2$, each such torus intersects the cylinder $r = R/2$ in two circles. We continue such a torus inside the cylinder by use of the "conical" surfaces

$$z^2 = x^2 + y^2 + C \tag{13}$$

for suitably chosen C.

An outward directed normal vector to such a surface is

$$\mathbf{n} = \begin{cases} \boldsymbol{\rho} - \dfrac{2G}{\rho} \sin\phi \cos\phi \sin\theta \, \boldsymbol{\phi} - \dfrac{G}{r} \sin^2\phi \cos\theta \, \boldsymbol{\theta} & (r > R/2) \\ z\mathbf{z} - x\mathbf{x} - y\mathbf{y} & (r < R/2) \end{cases}$$

where $\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi}$ $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ are mutually orthogonal unit vectors in the ρ, θ, ϕ (x, y, z) directions, respectively. The velocity vector $\mathbf{v}$ of a point moving along an orbit is

$$\mathbf{v} = \begin{cases} \dot{\rho}\boldsymbol{\rho} + r\dot{\theta}\boldsymbol{\theta} + \rho\dot{\phi}\boldsymbol{\phi} & (r \geqslant R/2) \\ \dot{x}\mathbf{x} + \dot{y}\mathbf{y} + \dot{z}\mathbf{z} & (r \leqslant R/2). \end{cases}$$

It is easy to compute the scalar product and get

$$\mathbf{n} \cdot \mathbf{v} = \begin{cases} -\dfrac{z^2 F}{\rho}\left(1 - \dfrac{2G}{\rho} \sin\phi \cos^2\phi \sin\theta\right) & (r > R/2) \\ -z^2 & (r < R/2). \end{cases} \tag{14}$$

This equation shows that (a) orbits inside $S = S(\varepsilon_0)$ pierce tori in an outward sense, for $\mathbf{n} \cdot \mathbf{v} \geqslant 0$ (recall that $F < 0$ inside S), (b) S is orbit-containing, for $\mathbf{n} \cdot \mathbf{v} = 0$ on S, and (c) orbits outside S pierce surfaces in an inward sense, so that they tend either toward S or else toward the critical point at the origin (note that they cannot approach other points in the plane $z = 0$, for they cannot stay in that plane and once they leave it they start piercing surfaces).

The circle $r = R$, $z = 0$ is itself an orbit; it is a completely unstable limit cycle.

We now investigate the critical point at the origin. For $r \leqslant R/2$, the differential equation is linear, with characteristic polynomial

$$P(s) = s^3 + s^2 + 1.$$

Since $P(-1) = 1$ and $P(-2) = -3$, there is a negative real root $-2 < \alpha < -1$. Since $\alpha + \beta + \gamma = -1$, the two complex roots β and γ have positive real parts. The origin is thus an unstable critical point. The two orbits corresponding to the functional form $x(t) = \pm e^{\alpha t}$ tend to the origin as time increases, and all other orbits outside S tend to S as time tends to infinity.

IV. Orbits in S

We have seen that the surface S contains orbits. We now show that there exists a value of δ such that these orbits are not periodic, and dense in S. This fact implies that each orbit approaches the entire surface S, not just part of it (see, e.g., Theorem 16.1.2 of Coddington and Levinson [1], the proof of which is valid for arbitrary dimension).

It is convenient to use only the two variables θ and ϕ to describe a point on S. Since $\dot{\theta}$ is always positive on S by (7a), we may take θ as our independent variable, and consider the differential equation

$$\frac{d\phi(\theta)}{d\theta} = \frac{\dot{\phi}}{\dot{\theta}} = \frac{\sin\theta + \dot{\theta}\,(\delta/\rho)\sin\phi\cos\phi\cos\theta}{\dot{\theta}(1 - (2\delta/\rho)\cos^2\phi\sin\theta)} = f(\theta, \phi) \tag{15}$$

where $\rho(\theta, \phi)$ is given by (9) and then $\dot{\theta}(\theta, \phi)$ is given by (7a).

THEOREM. *If one orbit in S is periodic, all are. If one orbit is not, all are not. If the orbits in S are not periodic, then each is almost periodic and dense in S.*

PROOF. This Theorem is proved in Theorems 17.3.2 and 17.4.2 of Coddington and Levinson [1]. The hypotheses of Theorem 17.4.2 are easily verified. In particular, the bounded variation of $\partial f/\partial\phi$ follows from the fact that it has only a finite number of finite maxima and minima.

By this Theorem, we must only show that there exists a δ such that one solution of (15) is not periodic.

We first observe that if $\delta = 0$, then each solution of (15) is periodic. For then, since $\dot{\theta}$ is even in θ by (7a), $f(\theta, \phi)$ is odd in θ and hence $\phi(\theta)$ is even in θ so that $\phi(-\pi) = \phi(\pi)$.

Let C denote the circle in S for which $\theta = 0, 0 \leqslant \phi \leqslant 2\pi$. If $\phi(\theta)$ denotes the solution of (15) through a point p of C, so that $(0, \phi(0)) = p$, define

$$u_p(\delta) = \phi(2\pi) - \phi(0),$$

so that u_p describes the net spiraling of the orbit as it circulates once around the torus S.

LEMMA 1. *If ε_0 is chosen small enough, there exists a range $-\delta_0 \leqslant \delta \leqslant \delta_0$ in which*

$$\frac{\partial u_p(\delta)}{\partial\delta} > 0$$

holds for each p.

PROOF. We will show that $u_p(\delta)$ can be expanded in a power series

$$u_p(\delta) = \sum_{i,j=0}^{\infty} a_{ij}\,\alpha^i\,\beta^j \tag{16}$$

where $\alpha = \delta/\varepsilon_0$ and $\beta = \varepsilon_0/R$, and that a_{10} is positive. Since

$$\frac{\partial u_p(\delta)}{\partial \alpha} = a_{10} + O(\beta)$$

this will suffice to prove the Lemma. It will be clear from the following detailed discussion of how to expand $u_p(\delta)$ (a discussion necessary to prove that $a_{10} > 0$) that the series exists and converges if first β and then α are chosen small enough.

Using the formula $(1 - x)^{-1} = 1 + x + x^2 + \ldots$ repeatedly, expand the expression (15) for $\phi'(\theta)$ into a double power series in α and β:

$$\phi'(\theta) = \sin\theta + \alpha(2\cos^2\phi \sin^2\theta + \sin\phi\cos\phi\cos\theta) + \ldots . \tag{17}$$

Since $\phi'(\theta)$ is analytic in α and β, so is $\phi(\theta)$ (for a proof, see Birkhoff [6, Section I.5]). Write $\phi(\theta) = c - \cos\theta + e(\theta)$ where $c = 1 + \phi(0)$ and use (17) to see that the coefficient of $\alpha^0\beta^0$ in the expansion for $e'(\theta)$, and hence in the expansion for $e(\theta)$, vanishes. Inserting this expansion for $\phi(\theta)$ into (17) gives, by using various trigonometric identities and expanding $\sin e(\theta)$ and $\cos e(\theta)$ in series, that

$$\begin{aligned}\phi'(\theta) = \sin\theta + \frac{\alpha}{2}\{[1 &+ \cos 2c\cos(2\cos\theta) \\ &+ \sin 2c\sin(2\cos\theta)](1 - \cos 2\theta) \\ &+ [\sin 2c\cos(2\cos\theta) - \cos 2c\sin(2\cos\theta)]\cos\theta\} \\ &+ \ldots .\end{aligned} \tag{18}$$

Since

$$u_p(\delta) = \int_0^{2\pi} \phi'(\theta)\, d\theta,$$

we simply integrate the coefficient of α in (18) from zero to 2π (use, say, Watson [7, pages 17—19]) to get $a_{10} = \pi$. The Lemma is proved.

LEMMA 2. *Let p be a point of C. For all but a countable number of δ's in $-\delta_0 \leqslant \delta \leqslant \delta_0$, the orbit γ_p through p never closes.*

PROOF. If γ_p is periodic, let $2\pi N(\delta)$ denote the increase in θ before γ_p closes. Let $N(\delta) = \infty$ if γ_p is not periodic.

We first show that if $N(\delta_1)$ is finite and M is positive, then there exists an open neighborhood d of δ_1 such that δ in d, $\delta \neq \delta_1$ imply $N(\delta) > M$. To see this, consider the successive points

$$p = p_0, p_1, p_2, \ldots, p_{N(\delta_1)-1} = p$$

of intersection of γ_p and C. If δ is increased slightly, these points move slightly toward larger ϕ, by Lemma 1. Since the points have a finite separation between them, it is clear that we may increase δ by such a small amount that $N(\delta)$ will exceed M. Similarly, if δ is decreased slightly from δ_1, all the points move toward smaller ϕ.

Now consider, for fixed N, the set D_N of δ's such that $N(\delta) < N$. About each δ in D_N there is a neighborhood in which $N(\delta) > N$. Since $(-\delta_0, \delta_0)$ is a finite interval, there are only a finite number of such neighborhoods whose length exceeds $1/m$, for each $m = 1, 2, 3, \ldots$. Hence there are only a countable number of such neighborhoods; but one is associated with each δ, so D_N is a countable set.

Let D denote the union of the D_N's over $N = 1, 2, \ldots$. Since D is a countable union of countable sets, D is itself countable. The compliment $\bar{D}$ of D in $(-\delta_0, \delta_0)$ is uncountable, and for each δ in $\bar{D}$, we have $N(\delta) = \infty$.

The author wishes to thank Professor R. Kronauer for many stimulating discussions.

References

1. Coddington, E. A. and Levinson, N., "Theory of Ordinary Differential Equations." McGraw-Hill, New York, 1955.
2. Friedrichs, K. O., On non-linear vibrations of third order. *In* "Studies in Non-linear Vibration Theory." Institute for Mathematics and Mechanics, New York University, 1946.
3. Levinson, N., Friedrichs, K. O., LeCorbeiller, P., and Stoker, J. J., Non Linear Mechanics, Brown University, 1942–43. A set of mimeographed lecture notes.
4. Rauch, L. L., Oscillation of a third order nonlinear autonomous system. *In* "Contributions to the Theory of Nonlinear Oscillations." Princeton Univ. Press, Princeton, New Jersey, 1950.
5. Levinson, N., A second order differential equation with singular solutions. *Ann. Math.* **50**, 127–153 (1949).
6. Birkhoff, G. D., "Dynamical Systems." Am. Math. Soc. Colloquium Publications, Vol. 9, 1927.
7. Watson, G. N., "Theory of Bessel Functions," 2nd ed., Cambridge Univ. Press, Cambridge, 1952.